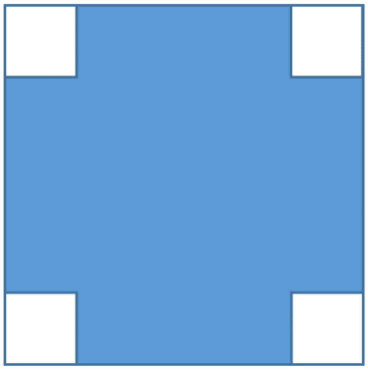
CSMC CIMC Prep 8

Warmup #1) A 5cm by 5cm square has its corners cut out in 1cm by 1cm. What is the largest square that can fit into the remaining shape? (AMC 8, 2015)



Question#2: From a jar of candies, Autumn takes “a” percent of the candies plus a more candies. Brooke takes “b” percent of the remaining candies plus “b” more candies. Here, “a” and “b” are positive integers less than 100. If Autumn and Brooke have taken the same number of candies, determine all possible values of “a” and “b” (COMC POW 2013)

Question #3) Evaluate the following and find the value of $n+k$, where “n” is as big as possible: (Both “n” and “k” are integers)

$$100C_1 + 2(100C_2) + 3(100C_3) + 4(100C_4) + ... + 100(100C_{100}) = k \times 2^n$$

Question #4) CSMC 2013

(a) Expand and simplify fully the expression $(a-1)(6a^2-a-1)$.

(b) Determine all values of θ with $6\cos^3\theta - 7\cos^2\theta + 1 = 0$ and $-180° < \theta < 180°$
Round each answer to 1 decimal place where appropriate.
(Note that $\cos^3\theta = (\cos\theta)^3$.)

(c) Determine all values of θ with $6\cos^3\theta - 7\cos^2\theta + 1 < 0$ and $-180° < \theta < 180°$

Question#5: Expressed the sequence below as a fraction in lowest terms: (Brilliant.org)

$$\frac{1\times 2}{10}+\frac{2\times 3}{10^2}+\frac{3\times 4}{10^3}+\frac{4\times 5}{10^4}+\ldots$$

1. 15   2. (a,b)=(20,25)   3 25×2^{101} → n+k=126   5. $\dfrac{200}{729} --> a+b=929$

4.a) $(a-1)(6a^2-a-1) = 6a^3 - a^2 - a - 6a^2 + a + 1 = 6a^3 - 7a^2 + 1$

b) Therefore, the solutions to the equation $6\cos^3\theta - 7\cos^2\theta + 1 = 0$, rounded to one decimal place as appropriate, are $0°, 60°, -60°, 109.5°, -109.5°$.

From this analysis, the values of θ for which $6\cos^3\theta - 7\cos^2\theta + 1 < 0$ are

c) $-180° < \theta < -109.5°$ and $-60° < \theta < 0°$ and $0° < \theta < 60°$ and $109.5° < \theta < 180°$

Question #2)

Let the total number of candies be n initially. After Autumn has taken $\frac{na}{100} + a$ candies, Brooke will take $(n - \frac{na}{100} - a) \cdot \frac{b}{100} + b$ candies. Equating these two expressions and simplifying, we obtain $na + 100a = nb - \frac{nab}{100} - ab + 100b$. This may be rewritten as $(n+100)(b - a - \frac{ab}{100}) = 0$. Since $n \neq -100$, the second factor must be zero so that $b = \frac{100a}{100-a}$. Note that $a < 50$ so that $100 - a > 50$. Let p be any prime divisor of $100 - a$. Then p must divide $100a$ so that it divides either 100 or a. It follows that p must divide 100, so that p can only be 2 or 5. This means that $100 - a = 64$ or 80. However 36 does not divide 6400. Hence $(a,b) = (20,25)$ is the only possibility. This does work if we take $n = 5k$ where $3k > 40$. Autumn will take $k + 20$ candies, leaving behind $4k - 20$. Then Brooke will take $(k-5) + 25 = k + 20$ candies.

Q3)

$100C_1 + 2(100C_2) + 3(100C_3) + \cdots$

$100 + 2\left(\frac{100\times 99}{1\times 2}\right) + 3\left(\frac{100\times 99\times 98}{1\times 2\times 3}\right) + 4\left(\frac{100\times 99\times 98\times 97}{1\times 2\times 3\times 4}\right) + \cdots\ 100\left(\frac{100}{1}\times\frac{99}{2}\times\cdots\frac{2}{99}\times\frac{1}{100}\right)$ FACTOR OUT THE "100"

$100\left(1 + \frac{99}{1} + \frac{99\times 98}{1\times 2} + \frac{99\times 98\times 97}{1\times 2\times 3} + \frac{99\times 98\times 97\times 96}{1\times 2\times 3\times 4} + \cdots\cdots \frac{99\times 98\times 97\times\cdots 1}{1\times 2\times 3\times 4\times\cdots 99}\right)$

$100\underbrace{\left(99C_0 + 99C_1 + 99C_2 + 99C_3 + 99C_4 + \cdots\cdots 99C_{99}\right)}_{2^{99}}.$

2^{99} (use Pascal's Triangle)

$1 = 2^0$
$1\ \ 1 = 2^1$
$1\ \ 2\ \ 1 = 2^2$
$1\ \ 3\ \ 3\ \ 1 = 2^3$
$1\ \ 4\ \ 6\ \ 4\ \ 1 = 2^4$
$1\ \ 5\ \ 10\ \ 10\ \ 5\ \ 1 = 2^5$

$0C_0 = 2^0$
$1C_0\ \ 1C_1 = 2^1$
$2C_0\ \ 2C_1\ \ 2C_2 = 2^2$
$3C_0 + 3C_1 + 3C_2 + 3C_3 = 2^3$
$\vdots$
$99C_0 + 99C_1 + 99C_2 + \cdots\cdots 99C_{99} = 2^{99}$

$\therefore$ the sum will be 100×2^{99} or $25\times 2^2\times 2^{99}$
$\boxed{= 25\times 2^{101}}$

Note that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$. From this fact, we do some algebra:

$$\binom{100}{1} + 2\binom{100}{2} + \ldots + 100\binom{100}{100} = \sum_{k=1}^{100} k\binom{100}{k} = \sum_{k=1}^{100} k\,\frac{100!}{(100-k)!k!}$$

$$= 100\sum_{k=1}^{n} \frac{(100-1)!}{(100-k)!(k-1)!} = 100\sum_{k=1}^{n}\binom{99}{k-1} = 100\cdot 2^{99} = 25\cdot 2^{101}.$$

So our final answer must be $25 + 101 = \boxed{126}$.

Q5)

A non-calculus approach,

Let, $S = \frac{1\times 2}{10} + \frac{2\times 3}{10^2} + \frac{3\times 4}{10^3} + \frac{4\times 5}{10^4} \dots\dots\dots\dots\dots\dots\dots\infty \qquad -------- \qquad (1)$

$\Rightarrow \frac{S}{10} = \frac{1\times 2}{10^2} + \frac{2\times 3}{10^3} + \frac{3\times 4}{10^4} + \frac{4\times 5}{10^5} \dots\dots\dots\dots\dots\dots\dots\infty \qquad ---------(2)$

Subtracting (1) *from* (2),

$\frac{9S}{10} = \frac{1\times 2}{10} + \frac{2\times 2}{10^2} + \frac{3\times 2}{10^3} + \frac{4\times 2}{10^4} \dots\dots\dots\dots\dots\dots\dots\infty$

$\Rightarrow \frac{9S}{20} = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{4}{10^4} \dots\dots\dots\dots\dots\dots\dots\infty \qquad ---------(3)$

$\Rightarrow \frac{9S}{200} = \frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \frac{4}{10^5} \dots\dots\dots\dots\dots\dots\dots\infty \qquad ---------(4)$

Again Subtracting (3) *from* (4),

$\frac{81S}{200} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} \dots\dots\dots\dots\dots\dots\infty \quad = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{1}{9}$

$\therefore \quad S = \frac{200}{729} = \frac{A}{B}$

Therefore $A + B = \boxed{929}$

$$\text{We know that} \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots \qquad \text{for } -1 < x < 1$$

$$\text{Differentiate both sides} \qquad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

$$\text{Differentiate again} \qquad \frac{2}{(1-x)^3} = 1\times 2 + 2\times 3x + 3\times 4x^2 + 4\times 5x^3 + \dots$$

$$\text{Multiplied by } x \qquad \frac{2x}{(1-x)^3} = 1\times 2x + 2\times 3x^2 + 3\times 4x^3 + 4\times 5x^4 + \dots$$

$$\text{Let } x = \frac{1}{10} < 1 \qquad \frac{\frac{2}{10}}{\left(1-\frac{1}{10}\right)^3} = \frac{1\times 2}{10} + \frac{2\times 3}{10^2} + \frac{3\times 4}{10^3} + \frac{4\times 5}{10^4} + \dots$$

$$= \frac{200}{729}$$

$\Rightarrow A + B = \boxed{929}$